

# Superfluid density of a pseudogapped superconductor near SIT

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We analyze critical behavior of superfluid density  $\rho_s$  in strongly disordered superconductors near superconductor-insulator transition and compare it with the behavior of the spectral gap  $\Delta$  for collective excitations. We show that in contrast to conventional superconductors, the superconductors with preformed pairs display unusual scaling relation  $\rho_s \propto \Delta^2$  close to superconductor-insulator transition. This relation have been reported in very recent experiments.

In a disordered conductor the superconductivity is first suppressed and then completely destroyed at high disorder. There are two mechanisms for the suppression (and eventually the full destruction) of the superconductivity by disorder (for recent reviews see<sup>1</sup> and<sup>2</sup>, chapter 1). The first mechanism attributes the suppression of the superconductivity to the increase of the Coulomb interaction that results in the decrease of the attraction between electrons and their eventual depairing.<sup>3</sup> In this mechanism the state formed upon the destruction of the superconductor is essentially a poor conductor. The alternative mechanism attributes superconductivity suppression to the localization of Cooper pairs that remain intact even when superconductivity is completely suppressed. The latter mechanism is called bosonic and the former fermionic. The theory of the bosonic mechanism has a long history: this scenario of the superconductor-insulator transition was suggested long ago<sup>4-7</sup> but was not developed further until recently<sup>2,8</sup> when experimental data<sup>9-13</sup> indicated the existence of a few materials that show such behavior. The bosonic mechanism is also supported by numerical computations<sup>14</sup>.

The interest to the superconductor-insulator transition without Cooper pair destruction is both fundamental and practical. First, it provides a perfect example of the disorder driven quantum transition in the closed system. Second, the bosonic superconductor in the vicinity of the transition might be ideal element for the isolation of the coherent quantum system from the environment<sup>15</sup> and for sensitive detectors of microwave radiation<sup>16</sup>. One of the most important properties for these applications is the value of the superfluid density of the superconductor. The goal of this paper is to compute this quantity in the bosonic mechanism; we also show that its behavior as the superconductivity is suppressed distinguishes fermionic and bosonic mechanisms of the superconductivity suppression.

Characteristic feature of the fermionic mechanism is that both the transition temperature  $T_c$  and the spectral gap  $\Delta$  at  $T \ll T_c$  are suppressed simultaneously, so that their ratio is left nearly constant in the broad range of  $T_c$  variation. At all disorders the superconductor remains qualitatively similar to a conventional BCS superconductor. When the superconductivity is completely suppressed by disorder the normal metal is formed, per-

haps with a weak tendency towards localization.

In contrast the state formed in the bosonic mechanism is qualitatively different from both the normal metal and conventional superconductor. In this mechanism the superconductivity disappears at the Superconductor-Insulator transition (SIT) whilst the gap in the single electron spectrum remains intact<sup>17</sup>. Superconducting state formed in the vicinity of SIT is therefore a pseudogapped state<sup>2</sup>. The best example of such behavior is amorphous InO films in which the critical temperature decreases by more than a factor of three while the single-particle gap stays practically independent on  $T_c$  but fluctuates spatially ( $\Delta_1 \approx 0.45 - 0.6$  meV)<sup>17</sup>. Another evidence of the bosonic mechanism is formation of a strong insulator characterized by a nearly-activated dependence of resistivity  $R(T)$  as observed in<sup>10,13</sup>.

The bosonic mechanism attributes the suppression of the superconductivity to the competition between Anderson localization and Cooper attraction between electrons assuming that the Coulomb interaction plays a minor role. The key feature of this mechanism is that the superconducting state is formed by the electrons in the *localized* single-electron eigenstates  $\psi_i(\mathbf{r})$ , with a relatively large localization length  $L_{loc}$  which depends on the proximity of the Fermi-energy to the Anderson mobility edge  $E_c$ . The presence of a length scale  $L_{loc}$  that is not related to superconducting properties of the conductor leads to the appearance of new energy scale in the problem that differs from the single electron gap in a usual BCS superconductor. Namely, attraction between two electrons that populate the same orbital state  $\psi_i(\mathbf{r})$ , leads to the formation of a bound pair with a binding energy

$$2\Delta_1 = g \int d^3r \psi_i^4(\mathbf{r}) \sim g L_{loc}^{-d_2}$$

where  $g$  is the Cooper attraction constant (with dimensionality [Energy]×[Volume]), and  $d_2 \approx 1.3$  is the fractal dimension of 3D Anderson transition, see<sup>2</sup>.

The energy scale  $\Delta_1$  discriminates against the odd (single electron population) while even (zero or two electrons) parts of the Hilbert space remain at low energies. As a result, single-particle density of states (DoS) acquires a pseudogap irrespectively of the presence of superconducting correlations. Experimentally, the appearance of the pseudogap in the absence of long range superconducting

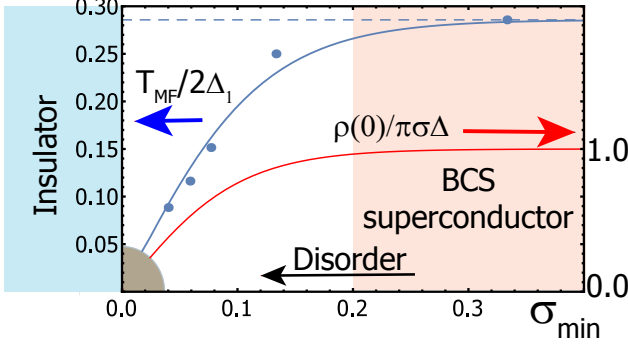


Figure 1. Sketch of the physical properties of the superconductors close to the disorder driven superconductor-insulator transition. As the disorder is increased the transition temperature drops while single particle gap remains constant resulting in the deviations from BCS relation  $2\Delta_1/T_{MF} \approx 3.5$  (indicated by the dashed line). At the same time the superfluid density  $\rho(0)$  at zero temperature also drops with the respect to the predictions of the BCS theory as shown by thin (red) line. The convenient measure of the disorder is provided by the minimal conductance,  $\sigma_{min}$  above the transition temperature. To make the connection with the experimental situation we have included in this plot the actual experimental data from<sup>17</sup> for InO. For these points  $\sigma_{min}$  is measured in  $\text{m}\Omega^{-1}\text{cm}^{-1}$ . Very close to superconductor-insulator transition superfluid density might be controlled by non-trivial critical exponents.

order was demonstrated in  $\text{InO}^{17}$  where strong suppression of low-voltage tunneling conductance was found up to  $T \geq 2T_c$ . Similar data were reported for sufficiently disordered TiN films<sup>18</sup>. In a pseudogapped superconductor long-range correlations develop due to coupling between localized bound pairs, and lead to formation of another energy scale,  $\Delta$ , that is related to superconducting long range order. The appearance of this energy scale was observed recently by Andreev tunneling spectroscopy in  $\text{InO}_x$  samples<sup>19,20</sup> close to SIT with  $T_c \sim 1.5\text{K}$ . These data indicate the appearance of the collective energy scale  $\Delta < \Delta_1$  that obeys BCS-like temperature dependence while  $\Delta_1$  is almost temperature independent at  $T < T_c$ . Another demonstration of the presence of some smaller energy scale that lies within the single-particle gap is provided by optical spectroscopy of  $\text{InO}_x$  and NbN samples of various disorder.<sup>21</sup>

Thin superconducting films are known to possess one more important energy scale,  $\Theta$ , that characterizes the dependence of the free energy on the phase gradient:

$$F = \frac{1}{2}\Theta \int d^2(\nabla\phi)^2$$

The helicity modulus  $\Theta$  determines the location of the Berezinsky-Kosterlitz-Thouless (BKT) transition<sup>22</sup>, at the transition  $\Theta(T_{BKT}) = \frac{2}{\pi}T_{BKT}$ . Energy  $\Theta$  is related to the superfluid density  $\rho_s$  which determines supercurrent in the film of thickness  $d$  in presence of a vector

potential:  $\mathbf{j} = -\rho_s \mathbf{A}/c$ , namely  $\Theta = (\hbar/2e)^2 \rho_s \cdot d$ .

For conventional disordered superconductors there is a linear relation (at  $T \ll T_c$ ) between superfluid density  $\rho_s$  and the energy gap  $\Delta$ :

$$\rho_s = \frac{\pi\sigma\Delta}{\hbar} \quad (1)$$

where  $\sigma$  is the normal-state resistivity<sup>23</sup>. Although usually derived in the framework of the BCS theory this relation is more robust because it can be traced to the optical weight conservation. So, not surprisingly, it is valid also for the moderately disordered InOx films with  $T_c \approx 2.5 - 3.5\text{K}$ , as was found via the kinetic inductance measurements<sup>24</sup>. Magnetoresistance studies of these films, see<sup>25</sup>, as well as Andreev tunneling spectroscopy<sup>20</sup>, show expected properties of disordered BCS superconductors, namely: no pseudogap, Andreev gap that has the same value as the single-particle gap, finally, the destruction of superconductivity by magnetic field leads to a normal metal state without noticeable  $R(B)$  peak.

We now demonstrate that a pseudogapped superconducting state is characterized by a different relation between  $\Theta$  and  $\Delta$ , leading to a number of unique features. We start with the expression for the full spectral weight  $K^{tot}$  for frequency-dependent conductivity as derived in Sec. 6.4 of Ref.<sup>2</sup>:

$$K^{tot}(T) = \frac{e^2}{\hbar^2 \mathcal{V}} \sum_{ij} g M_{ij} x_{ij}^2 \frac{\Delta_i \Delta_j \tanh \beta \varepsilon_i \tanh \beta \varepsilon_j}{\varepsilon_i \varepsilon_j}. \quad (2)$$

Here  $\beta = 1/T$ ,  $\mathcal{V}$  is the system's volume,  $i, j$  enumerate single-electron eigenfunctions those eigenvalues are  $\xi_i, \xi_j$ , matrix elements  $M_{ij} = \int d^3r \psi_i^2(\mathbf{r}) \psi_j^2(\mathbf{r})$ , and  $\varepsilon_i = \sqrt{\xi_i^2 + \Delta_i^2}$ . The quantities  $\Delta_i$  are the order parameter amplitudes related to the superconducting order parameter in a real space  $\Delta(r)$  by

$$\Delta(\mathbf{r}) = \frac{g}{2} \sum_i \Delta_i \frac{\tanh(\beta \varepsilon_i)}{\varepsilon_i} \psi_i^2(\mathbf{r}) \quad (3)$$

In the mean-field approximation, the amplitudes  $\Delta_i$  obey self-consistency equations

$$\Delta_i = \frac{g}{2} \sum_j \Delta_j M_{ij} \frac{\tanh(\beta \varepsilon_j)}{\varepsilon_j} \quad (4)$$

Mean-field approximation of Ref.<sup>2</sup> assumes that amplitudes  $\Delta_i$  are actually a slow functions of the single-particle energies  $\xi_i$ , i.e. one can replace  $\Delta_i$  by a regular function  $\Delta(\xi)$  evaluated at  $\xi = \xi_i$ ; this function has to be determined from the continuous version of Eq.(4),

$$\Delta(\xi) = \frac{\lambda}{2} \int d\zeta M(\xi - \zeta) \frac{\tanh(\beta \varepsilon(\zeta))}{\varepsilon(\zeta)} \Delta(\zeta) \quad (5)$$

where  $\lambda = g\nu_0$  and  $\nu_0$  is the DoS value in the normal state (per single spin projection), and  $M(\omega) = \mathcal{V} \overline{M_{ij}}$  with  $|\xi_i - \xi_j| = \omega$ .

Eq.(2) was derived under the assumption that eigenfunctions  $\psi_{i,j}(\mathbf{r})$  that contribute mostly to the sum over  $i, j$  are relatively well-localized, with the typical distances  $x_{ij} \sim R_0$  between the maxima of their envelopes that are somewhat larger than localization length  $L_{loc}$ . We estimate effective interaction range as  $R_0 \sim L_{loc} \ln \frac{\delta}{\Delta}$ , where  $\delta_L = (\nu_0 L_{loc}^3)^{-1}$  is the level spacing within localization volume. Pseudogaped superconducting state is realized when  $\Delta < \Delta_1 \ll \delta_L$ , so that  $R_0 \gtrsim L_{loc}$ .

At low temperatures the optical sum rule arguments show that the major contribution to the total spectral weight  $K^{tot}$  comes from the superconducting density, i.e  $K^{tot}(0) \approx \rho_s(0) \equiv \rho_s$ .<sup>2</sup> The Eq.(2) can be simplified by eliminating the sum over  $j$  with the help of (4) and by replacing square of dipole matrix elements by its average,  $x_{ij}^2 \rightarrow R_0^2/2$ . At  $T = 0$  we get

$$\rho_s \approx \frac{e^2 R_0^2}{\hbar^2 \mathcal{V}} \sum_i \frac{\Delta_i^2}{\epsilon_i} = \frac{2\nu_0 e^2 R_0^2}{\hbar^2} \int_0^\infty \frac{d\xi \Delta^2(\xi)}{\sqrt{\xi^2 + \Delta^2(\xi)}} \quad (6)$$

$$\approx \frac{2\nu_0 e^2 R_0^2}{\hbar^2} \Delta^2 \quad (7)$$

We emphasize that  $\Delta$  in this equation represents *collective* gap as measured by the Andreev spectroscopy<sup>20</sup> and in the THz optical measurements<sup>21</sup>, which is smaller than single-particle gap  $\Delta_1$ .

The major difference between relations (7) and (1) is that in the pseudogapped superconductor  $\rho_s \sim \Delta^2$ , whereas in the usual case  $\rho_s \sim \Delta$ . Note that general arguments related to optical weight conservation are not applicable<sup>2</sup> for the pseudogapped superconductor due to the presence of the second energy scale,  $\Delta_1$ . The unusual scaling of  $\rho_s \propto \Delta^2$  in Eq.(7) is due to the presence of an independent spatial scale  $R_0$  that determines the range of the tunneling matrix elements  $M_{ij}$  between localized eigenstates  $\psi_i(\mathbf{r})$ . It is crucial that  $R_0$  weakly depends on  $\Delta$ . The counterpart of  $R_0$  in a usual dirty superconductor is given by its low-temperature coherence length  $\xi_0 \approx \sqrt{\hbar D/\Delta}$ . In this case instead of  $R_0^2$  one should use  $\xi_0^2 \propto D/\Delta$  and linear relation  $\rho_s \propto \Delta$  is restored. Note that quadratic scaling  $\rho_s \propto \Delta^2$  is known for superfluidity in Bose systems, see for example<sup>26</sup>.

The prediction of unusual scaling between  $\Delta$  and  $\rho_s$  is important close to SIT where both  $\Delta$  and  $\rho_s$  are expected to decrease strongly. Observation of the scaling  $\rho_s \propto \Delta^2$  would serve as an independent demonstration of the bosonic nature of a superconductive state.

The crucial step in the derivation of the result (7) is the averaging over  $\epsilon_i$  and  $\Delta_i$ . Close to the transition to the insulating state the superconducting order parameter becomes very inhomogeneous<sup>27</sup>, which requires reexamination of this procedure. In this regime distribution of the order parameter becomes very broad. In the Bethe lattice approximation it is given by

$$P(\Delta) \approx \frac{\Delta_0^m}{\Delta^{1+m}} \quad \Delta_0 \leq \Delta \leq \Delta_{max} \quad (8)$$

Here  $\Delta_0$  denotes the typical (most probable) value of the local order parameter  $\Delta$ , the exponent  $m$  is slightly less

than unity:  $1 - m = e\lambda \ll 1$  where  $e = 2.718...$ . Distribution (8) is applicable up to the upper cut-off  $\Delta_{max} \sim \Delta_1$ .

Another kind of the order parameter distribution was obtained numerically by Lemarie et al for 2D attractive Hubbard model with strong random potential<sup>28</sup>; it was claimed to be related to a distribution of the Tracy-Widom type<sup>29</sup> that do not have power law tails of the form (8), but does have a large dispersion.

In presence of strong statistical fluctuations of  $\Delta_i$  the MFA equations (4,5) are not valid, thus it is not possible to follow the route of calculations that lead to Eq.(7). Instead we note that original Eq.(2) contains double sum over sites  $i, j$  over large number  $\sim Z \gg 1$  of statistically independent terms for each  $i$ . Thus it is possible to estimate this sum by

$$\rho_s \sim \frac{\nu_0 e^2 R_0^2}{\hbar^2} \Delta_0^2 \quad (9)$$

where  $\Delta_0$  is the typical (most probable) value of the local order parameter  $\Delta_i$ , irrespectively of its specific distribution  $P(\Delta)$ . Qualitatively, one expects that rare pairs of states  $i, j$  with anomalously large  $\Delta_i \gg \Delta_0$  cannot contribute considerably to macroscopic superfluid density  $\rho_s$  since they will be "screened" by weaker typical pairs due to redistribution of the supercurrent density.

Notice that all quantities entering Eq.(9) except for the interaction range  $R_0$  are measurable. The latter can be determined if the scaling relation (9) is observed in a broad range of  $\Delta_0$  and  $\rho_s$ .

Prediction (9) is in a rough qualitative agreement the data that became available very recently<sup>21</sup>. Indeed, total evolution of the gap as measured in this experiments by optical spectroscopy, Fig.2c, spans about one order of magnitude between the "crossing point" corresponding to the reduced transition temperature  $\tilde{T}_c = 0.5$ , and the most disordered samples with  $\tilde{T}_c \approx 0.2$ . At the same time, superfluid density  $\rho_s$  (extracted from the imaginary part of the optical impedance) changes by nearly 2 orders of magnitude in the same range of  $\tilde{T}_c$ , according to Fig.3b of the same paper.

In the close vicinity of the quantum phase transition from superconducting to insulating state one expects critical behavior characterized by the exponents that might differ from the mean field result (9), especially in low dimensional systems. For instance, the recent numerical works<sup>30,31</sup> on the hard core boson model reported behavior  $\rho_s \propto \Delta^a$  with the exponent  $a \approx 2.5$  in the critical regime. However, applicability of 2D scaling for strongly disordered superconducting films studied experimentally is not evident, because many of these films are not thin enough to be considered two-dimensional.

Close to the critical point one expects large spatial fluctuations of the order parameter. The resulting fluctuations of the superfluid density were studied numerically in the recent paper<sup>30</sup> for the 2D hard-core boson model with random potential, using Quantum Monte Carlo method on systems with linear sizes in the interval  $L = 12 - 32$ . A broad probability distribution  $\mathcal{P}(\ln \rho_s)$  was found both

in superfluid and insulating phases; however, on superfluid side of the transition, the width of this distribution diminishes with  $L$ , whereas an opposite tendency is found for the insulating state. These results indicate that in a macroscopic system  $dc$  superfluid density is a self-averaging quantity in the superconducting state. Analytical calculation of the corresponding distribution  $\mathcal{P}(\ln \rho_s)$  in the framework of the theory developed here is an interesting problem which we leave for future studies.

The spatial fluctuations of  $\rho_s$  are probably observable in the low frequency measurements. Usually  $\rho_s$  is measured via  $ac$  kinetic inductance  $\mathcal{L} \propto 1/\rho_s$ , see for example<sup>32,33</sup>. Nonzero measurement frequency  $\omega$  determines the length scale  $l_\omega = \sqrt{D/\omega}$  where coherent transport takes place. With a typical diffusion constant for very strongly disordered superconductors  $D \sim 0.1 - 1 \text{ cm}^2/\text{s}$ , the length  $l_\omega$  ranges from  $0.02 - 0.5$  micron for high-frequency measurements<sup>32</sup> to  $10 - 100$  micron for low-

frequency ones<sup>33</sup>. In the latter case it might be possible to detect spatial fluctuations of  $\rho_s$  with a scanning technique.

Sharp drop near SIT of the helicity modulus  $\Theta = (\hbar^2/4e^2)\rho_s \cdot d$  according to (9) leads to an enhancement of both thermal and quantum phase fluctuations, and to suppression of both  $T_c$  and critical magnetic field  $H_c$  with respect to the mean-field estimates, see also Ref.<sup>34</sup>.

In conclusion we have shown that the bosonic mechanism of the superconductor-insulator transition implies a nearly quadratic scaling of the superfluid density with the superconducting order parameter, the prediction that can be verified experimentally. This resulting very low values of the superfluid density would make materials close to the transition extremely useful for the applications.

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